Three structural properties reflecting the synchronizability of complex networks

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During the process of adding links, we find that the synchronizability of the classical Barabási-Albert (BA) scale-free or Watts-Strogatz (WS) small-world networks can be statistically quantified by three essentially structural quantities of these networks, i.e., the eccentricity, variance of the degree distribution, and clustering coefficients. The results indicate that both the eccentricity and clustering coefficient are positively linearly correlated with synchronizability, while the variance is negatively linearly correlated. Moreover, the efficiency of some particular strategies of adding links to change the synchronizability is also investigated. This information can be used to guide us to design corresponding strategies of structure-evolving processes to manipulate the synchronizability of a given network.

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Two typical and important networks are the small-world (SW) networks [1,2] and scale-free (SF) networks [3]. Many more real-world networks [4–6], including both artificial and natural systems, seem to hold either the SW property or the SF property or both of them.

In recent years, the research on synchronizability for complex networks has become an important and interesting topic [7–10] (and many references cited therein). These study results imply that the structural properties of a network have a close relationship with its synchronizability; but some conclusions from them seem somewhat inconsistent. For example, for better synchronization, a small value of the maximal betweenness centrality is required [8], rather than short characteristic path length or large heterogeneity of the degree distribution. While in [11] the authors strongly suggested that the maximal betweenness may not give a comprehensive description of network synchronizability. A more interesting investigation [12] has been given to show that two simple graphs, which have the same structural quantities such as average distance, degree distribution, and node betweenness centrality, have very different synchronizabilities. From these research results, we can see that it must be unreasonable to study the synchronizability of a given network just by investigating its some structural quantities.

In spite of this, however, we will show that the synchronizability of a given class of network, during a structure-evolving process of adding a fraction of new links, can be statistically quantified by their some essentially structural quantities. Then, some essential and important problems appear; for example, (i) which structural properties play a crucial role in the synchronizability of a given class of network during such structure-evolving process?, (ii) to what extent do these crucial structure quantities affect synchronizability during such process?, and (iii) how to design an effective strategy for such process such that the synchronizability can be improved or weakened? In this Brief Report, by using the important classes of Barabási-Albert (BA) scale-free and Watts-Strogatz (WS) small-world network models, we address these problems with slight structure transformations performed by adding a fraction of links to these networks.

First, we present the general framework of synchronizability established in [13], which we follow for the sake of simplicity. The state equation of considered network has the following form:

$$\dot{x}_i(t) = F(x_i(t)) + c\sum_{j=1}^{N} a_{ij}H(x_j(t)), \quad i = 1,2, \ldots ,N, \quad (1)$$

where the function $F(\cdot)$ describes the local dynamics of nodes and $H(\cdot)$ represents the inner-coupling rule. The constant $c>0$ denotes the overall strength of coupling and $\mathbf{A}=(a_{ij}) \in \mathbb{R}^{N\times N}$ with zero-sum rows shows the coupling configuration of the network. If nodes i and j are connected then $a_{ij}=a_{ji}=-1$; otherwise $a_{ij}=a_{ji}=0$. If node i is connected directly to k_i other nodes then $a_{ii}=k_i$. The linear stability of the synchronized state $\{x_i(t)=s(t), \forall i\}$ is determined by $N-1$ variational equations in the transverse directions, which have the uniform expression $\dot{\eta}=[DF(s)+\lambda DH(s)]\eta$ obtained by a diagonalization process. Let $0=\lambda_1<\lambda_2\leq \cdots \leq \lambda_N$ be the eigenvalues of $\mathbf{A}$. The synchronizability can be quantified by the eigenratio $R=\lambda_N/\lambda_2$ [13]: the smaller the eigenratio is the stronger synchronizability of the network (1).

In the following, we adopt a statistical method to discover the essential forces affecting the synchronizability caused by some crucial and typical structural quantities in the structure-evolving process performed by adding links. These quantities are selected primarily by the following scheme. First, since there are so many quantities to describe properties of a network, we only consider some typical quantities such as the average path length, clustering coefficient, maximal betweenness centrality, eccentricity, variance of the degree distribution, and so on. Second, we view those quantities among which there exist strongly positive correlations in the process of adding links as a class and choose one from them randomly. Finally, these selected quantities should change smoothly in the process of completely random addition of links even for relatively small times of realization. Under this scheme and a large amount of simulations, we pick out the eccentricity (E), variance of the degree distribution (V),

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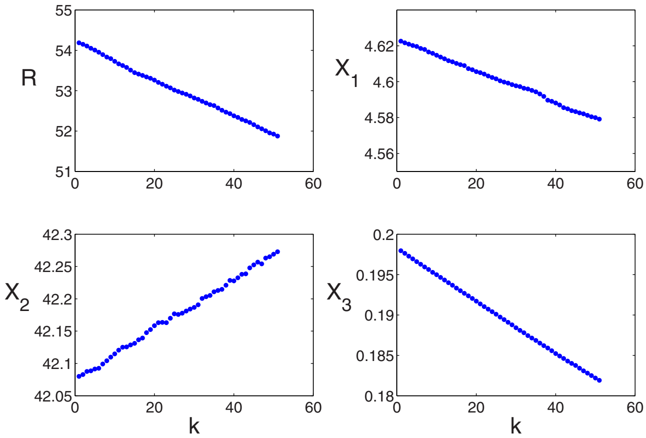

FIG. 1. (Color online) The approximately linear relations between $R(k)$ and $X_i(k)$, $i=1,2,3$.

and clustering coefficient (C) as the primary quantities for further consideration. Certainly, with the above rough procedure the selection of such quantities is not unique; but, we will show that with only these three quantities the synchronizability can be well quantified with just a fraction of new links added.

The eccentricity of a network is defined as $E=(\sum_{i=1}^{N}E_i)/N$, where $E_i=\max_{j=1,j\neq i}^{N}\{d_{ij}\}$ is the eccentricity of the node i and d_{ij} is the length of a shortest path between nodes i and j. The variance $V=\langle N^{-1}\sum_i k_i^2\rangle-\langle (N^{-1}\sum_i k_i)^2\rangle$ can quantify the heterogeneity of the degree distribution. The clustering coefficient is usually defined [14] as $C=(\sum_i C_i)/N$, where $C_i=2e_i/[k_i(k_i-1)]$, and e_i is the number of links that exist between the k_i neighbors of node i. For simplicity, these three structural quantities just mentioned above are denoted in turn by $X_i(k)\in\mathbb{R}$, $i=1,2,3$, and $k=1,2,\ldots,M$ that are random variables because of the randomicity in the following strategies of adding links, where M is the maximal number of added links. By plentiful simulations with adequately small M with respect to the number N_v of vacant links (i.e., the ratio $M/N_v \ll 1$), we surprisingly find that there exist strongly linear correlations between R and X_i in some typical strategies of adding links, i.e., $R(k)\sim X_i(k)$. The N_v will be calculated later. Since some well-known analysis tools such as the perturbation of coupling matrices, graph operations, and so on do not easily deal with this complicated case by noting the approximately linear relations between $R(k)$ and $X_i(k)$, this induces us to apply the linear regression analysis to seek the quantitative relations between them.

All the following simulations are based on 100 independent realizations. With $N=500$ and $M=50$, Fig. 1 shows the mean changes of R and X_i in the process achieved by the completely random addition of links (denote completely random (CR) strategy) into the initial BA network models (see [14]). These initial BA networks ($M=0$) start with $m_0=4$ nodes and the number $m=3$ of links introduced by a new node at every time step. Here, we have $N_v\approx C_N^2-mN$ because about mN links already exist in the initial BA networks. With $N=500$ and $M=50$, we get the ratio $M/N_v\approx 5\times 10^{-4}\ll 1$.

By applying the ridge regression [15,16] and still using R

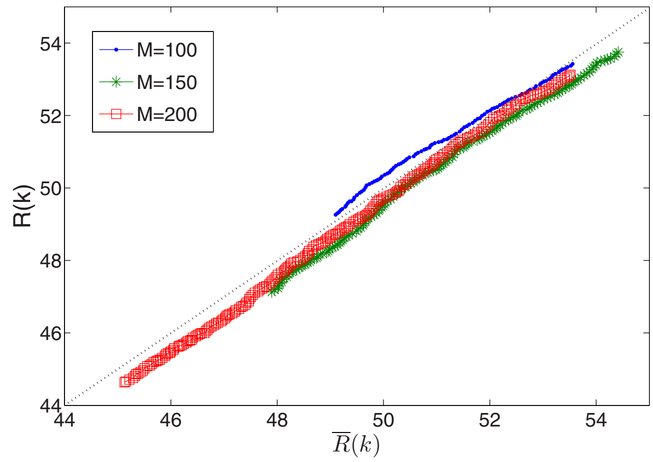

FIG. 2. (Color online) The differences between actual eigenratio $\bar{R}(k)$ with CR strategy and prediction eigenratio $R(k)$.

and X_i to denote these quantities after centering and scaling for comparability, we can obtain the regression equation

$$R(k)=0.199\,8X_1(k)-0.208\,2X_2(k)+0.266\,7X_3(k). \quad (2)$$

By other 100 independent realizations, we can always get $\{\sum_{k=1}^{M}[\bar{R}(k)-R(k)]^2\}/M\ll$ the variance of $\bar{R}$; here R is the prediction generated by the model (2) and $\bar{R}$ is the actual value. Figure 2 shows the differences between the actual eigenratio and prediction eigenratio that is calculated by using the regression (2), which displays an effective fitting. All the following constructed regression models are checked by this method for their validity and we omit the test processes for simplicity. From the coefficients on X_i in Eq. (2), more compactly denoted by a weight vector $w=(0.199\,8,-0.208\,2,0.266\,7)$, we can see that both the eccentricity and clustering coefficients play a positive correlation in synchronizability, while the variance plays a negative correlation. Here, we must point out that the regression model (2) is valid only for sufficiently small ratio M/N_v. For instance, with $M=500$ the approximately linear relations between R and X_i are destroyed (see from Fig. 3). So, the linear regression

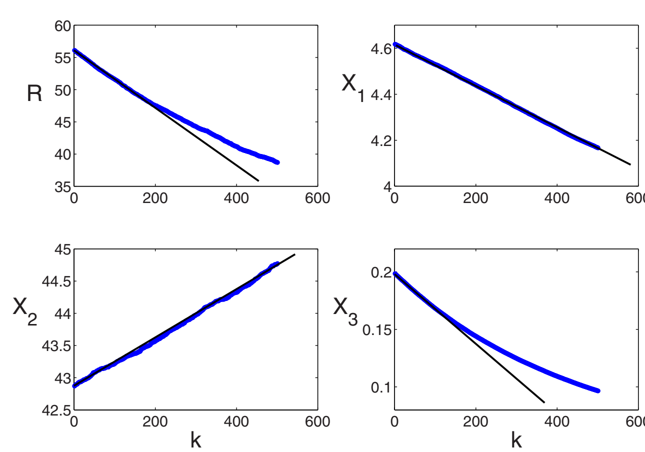

FIG. 3. (Color online) The changes of $R(k)$ and $X_i(k)$, $i=1,2,3$. The thin lines are all straight lines.

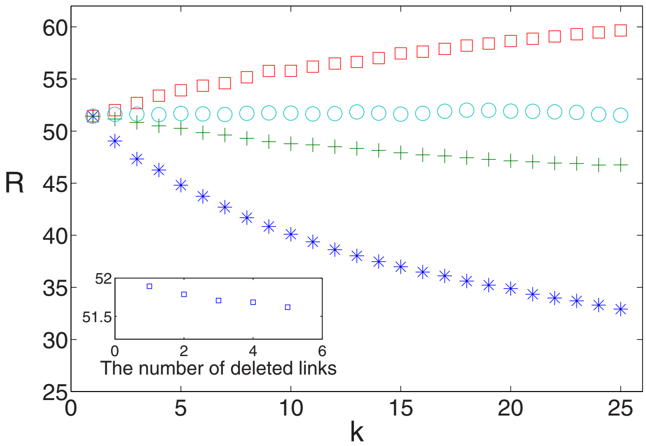

FIG. 4. (Color online) The changes of eigenratio R under strategies ME (star), SD (plus), LD (square), and SLD (circle) to add links into BA networks with $\delta=0.1$.

analysis (to seek the quantitative relations between them) becomes unsuitable.

In order to further verify that the weights on the corresponding quantities do show the strengths of force in synchronizability when new links are added, we design additional four specific strategies for adding links all different from the CR strategy. We call the first scheme the maximal eccentricity (ME) strategy: for each addition, we add a new link between the node chosen randomly with the *maximal eccentricity* and a node selected also randomly with the maximal distance to it. The second strategy is called the small degree (SD) strategy: for each addition, add a new link between two nodes randomly selected from those $[\delta N]$ nodes with relatively *small connectivity degrees*, whose number is quantified by the $\delta \in (0,0.5)$. Generally for small δ, relatively small M is needed for more effective simulations. The third one is large degree (LD) strategy: for each addition, add a new link between two nodes randomly selected from those $[\delta N]$ nodes with relatively *large degrees*. The last strategy we call the small and large degree (SLD) strategy: for each addition, add a new link between two nodes, one of which is chosen from those $[\delta N]$ nodes with relatively small degrees and the other one from those $[\delta N]$ nodes with relatively large degrees. Since any given strategy of adding links can be viewed as a combination of these five strategies, without loss of generality we only consider them for simplicity. Figure 4 shows the different change processes of R under these four strategies, which mean

$$\mathrm{ME} \succ \mathrm{SD} \succ \mathrm{SLD} \succ \mathrm{LD}, \tag{3}$$

for the same number of links added, where the notation $\succ$ represents stronger synchronizability between two strategies. Under each of the above four strategies, we still observe similar linear correlations between R and X_i as to the case of CR. By regressions, we can get four regression equations which are determined by the weights on the structural quantities that have been shown by Table I.

It is obvious that the CR method of adding links can be viewed statistically as an integration of various methods of adding links to a network. This induces us to consider the fluctuation of X_i caused by the four specific strategies based on the velocity quantified by the weight in Eq. (2) to see if that weight can accurately reveal the forces of the three structural quantities in synchronizability in the process of adding links. So, it is natural and valid to define $\beta_{\mathrm{ME}}=w_1w^T$, $\beta_{\mathrm{SD}}=w_2w^T$, $\beta_{\mathrm{LD}}=w_3w^T$, and $\beta_{\mathrm{SLD}}=w_4w^T$ as measures of the fluctuation based on the above analysis. By computation, we have $\beta_{\mathrm{ME}}=0.171\,3$, $\beta_{\mathrm{SD}}=0.061\,1$, $\beta_{\mathrm{LD}}=-0.016$, and $\beta_{\mathrm{SLD}}=0.051\,8$ that gives

TABLE I. The weights on the three structural quantities.

Strategy	X_1	X_2	X_3	Weight[a]
CR	0.1998	−0.2082	0.2667	w
ME	0.6557	0.483	0.5281	w_1
SD	0.2041	0.242	0.265	w_2
LD	−0.1066	0.1324	0.1232	w_3
SLD	0.0666	−0.0334	0.1181	w_4

[a]All data after centering and scaling.

$$\beta_{\mathrm{ME}} > \beta_{\mathrm{SD}} > \beta_{\mathrm{SLD}} > \beta_{\mathrm{LD}}, \tag{4}$$

which is consistent with Eq. (3). Moreover, the global trend of synchronizability under each of the four strategies can be estimated by the sign of w_iw^T. For example, $\beta_{\mathrm{SLD}}>0$ and $\beta_{\mathrm{LD}}<0$ means an increase in synchronizability under SLD and a decrease under LD (see Fig. 4). Therefore, we conclude that with an adequately small ratio M/N_v, the weight in Eq. (2) can show the forces statistically of the corresponding quantities in synchronizability in the process of adding links. From this weight, we know that the clustering coefficient affects the synchronizability most strongly. Certainly, for all other scale-free networks with arbitrary values of degree decay exponent, the above-obtained results may be valid. Further analysis in this regard will be considered in the future.

Next, let us consider the WS small-world networks, where all the notations have the same meanings as the case for the above BA scale-free networks. These initial WS networks ($M=0$) all include $N=500$ nodes and are constructed with a uniform probability $p=0.1$ for rewiring links [14]. The change processes of eigenratio R under the specific strategies are presented in Fig. 5 with $M=25$ new links added. Similarly, by the linear regression analysis, we can also obtain five regression models which are determined by the weights on the three structural quantities presented in Table II. The weight vector w also implies that both the eccentricity and clustering coefficients are positively linearly correlated with synchronizability, while the variance is negatively linearly correlated. Then from Table II, we get $\beta_{\mathrm{ME}}=3.936\,7$, $\beta_{\mathrm{SD}}=0.324\,3$, $\beta_{\mathrm{LD}}=-1.894\,6$, and $\beta_{\mathrm{SLD}}=-0.116\,9$ that gives

$$\beta_{\mathrm{ME}} > \beta_{\mathrm{SD}} > \beta_{\mathrm{SLD}} > \beta_{\mathrm{LD}}. \tag{5}$$

On the other hand, from Fig. 5, it is clear to see that

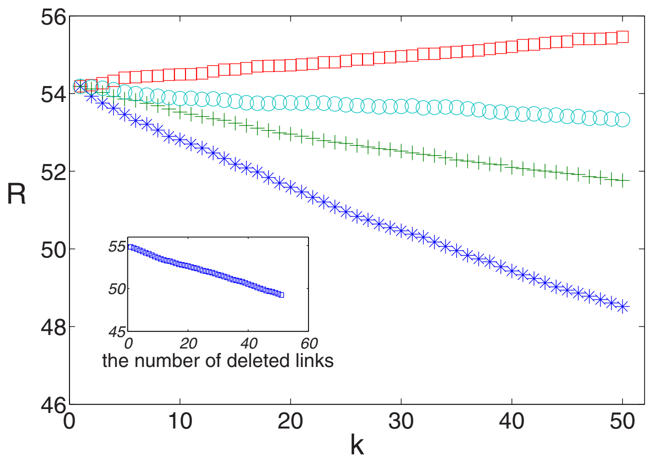

FIG. 5. (Color online) The changes of eigenratio R under strategies ME (star), SD (plus), LD (square), and SLD (circle) to add links into WS networks with $\delta=0.1$.

$$\mathrm{ME} > \mathrm{SD} > \mathrm{SLD} > \mathrm{LD}, \qquad (6)$$

which is consistent with the prediction (5). Moreover, the negative values of β_{LD} and β_{SLD} definitely indicate the weakening of synchronizability under these two strategies. So for WS networks, we also conclude that the weight w can reflect the forces of the corresponding quantities in synchronizability in the process of adding links.

From Figs. 4 and 5, we see that the LD strategy weakens the synchronizability both for BA and WS networks. In view of this, we conjecture that the synchronizability would be statistically strengthened if we delete a fraction of links among nodes with relatively large degrees. Considering the heterogeneous degree distribution in scale-free network and the homogeneous degree distribution in small-world network, we only delete 50 existing links in BA networks ($N=500$) and five links in WS networks ($N=500$). This has been verified by the insets of Figs. 4 and 5 based on 100 realizations. This case is consistent with the result from [9]. From Tables I and II, the weight vector w tells us that the ME strategy may be the most effective method to improve synchronizability for the BA and WS networks with a given small M, as the eccentricity and the clustering coefficients are greatly reduced, indicated by the weight w_1. But, with the addition of more new links (larger M), the predominance of the ME strategy to enhance the synchronizability seems to be weakened or even eliminated, as compared to other strategies.

TABLE II. The weights on the three structural quantities.

Strategy	X_1	X_2	X_3	Weight[a]
CR	0.9122	−0.3893	0.7531	w
ME	2.8459	−0.8572	1.3371	w_1
SD	0.3654	0.7429	0.3721	w_2
LD	−1.1552	0.2142	−1.0057	w_3
SLD	−0.1162	−0.1164	−0.0746	w_4

[a]All data after centering and scaling.

With different tuned parameters such as N, δ, and p, we can get similar results to those obtained above, but only as the added new links are just a fraction of vacant links, i.e., $M/N_v \ll 1$. Fortunately, for practical applications, this condition means that one would like to make a slight change to a network. Since the addition of links is just one method for structural transformation, one may apply similar techniques to consider the various influences of structural properties on synchronizability by other structural transformations such as deleting links, adding nodes or deleting nodes, and so on.

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[1] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[2] D. J. Watts, *Small Worlds* (Princeton University Press, Princeton, NJ, 1999).
[3] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[4] R. Albert *et al.*, Nature (London) **401**, 130 (1999).
[5] S. H. Strogatz, Nature (London) **410**, 268 (2001).
[6] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[7] T. Nishikawa, A. E. Motter, Y. C. Lai, and F. C. Hoppensteadt, Phys. Rev. Lett. **91**, 014101 (2003).
[8] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, Phys. Rev. E **69**, 067105 (2004).
[9] C. Y. Yin, W. X. Wang, G. Chen, and B. H. Wang, Phys. Rev. E **74**, 047102 (2006).
[10] C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).
[11] M. Zhao *et al.*, Physica A **371**, 773 (2006).
[12] Z. Duan, G. Chen, and L. Huang, Phys. Rev. E **76**, 056103 (2007).
[13] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998); K. S. Fink, G. Johnson, T. Carroll, D. Mar, and L. Pecora, Phys. Rev. E **61**, 5080 (2000); M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[14] X. F. Wang, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 885 (2002).
[15] A. E. Hoerl and R. W. Kennard, Technometrics **12**, 55 (1970).
[16] A. E. Hoerl and R. W. Kennard, Technometrics **12**, 69 (1970).